

A conducting ball in an axial electric field

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Abstract

We describe the distribution of a charge, the electric moments of arbitrary order and the force acting on a conducting ball on the axis of the axial electric field. We determine the full charge and the dipole moments of the first order for a conducting ball in an arbitrary inhomogeneous harmonic electric field. All statements are formulated in the form of theorems with proofs basing on properties of the matrix of moments of the Legendre polynomials. The analysis and proof of these properties are presented in Appendix.

1 Introduction

When a grounded conducting ball is installed on the axis of an external axial harmonic field, an induced charge arises on its surface. It shields the initial field so that the total potential of exterior and shielded fields inside the ball be equal to zero. If the potential of the ball with radius r is equal to U , then the distribution of the charge density on its surface will differ from that of a grounded ball on a constant, that is equal to $\varepsilon_0 U/r$, where ε_0 is electric constant.

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2 A CONDUCTING BALL IN AN AXIAL ELECTRIC FIELD

The density of a surface charge and the dipole moment of a ball in a homogeneous electric field are well known (see, for example, [1]). We propose the new efficient method for determining the surface charge density of a conducting ball in an inhomogeneous electric field. The required density is the solution of a one-dimensional integral equation, when the external field is a harmonic in the ball's volume. From here we obtain the analytical solution for the charge density in the case when the external field potential is a polynomial of arbitrary order on the axis of symmetry. The three theorems are proved. The first one consists of the two statements. The charge of the ball is fully determined by the value of potential of the external field at the center of the ball. The dipole moment of the ball is fully determined by the value of the external field intensity at the center of the ball. The second theorem is a generalization the first one and it declares that the dipole moment of $2k$ order of the ball is fully determined by the values of the first $k+1$ even derivatives from the potential at the ball's center, and the dipole moment of $2k+1$ order of the ball is fully determined by the values of the first $k+1$ odd derivatives from the potential at the ball's center. In the third theorem, we find the analytical form for the force acting on the ball when the axial potential of external field is a polynomial. In Appendix, we present and prove properties of the matrix of moments of the Legendre polynomials that are necessary for proving all the theorems. In particular, these are lemmas about linear dependence and orthogonality. We obtain explicit expressions for the entries of the inverse matrix. This circumstance allows one to find the surface charge density analytically.

The investigations presented in Appendix could be of special interest in theory of orthogonal polynomials. By this reason, formulas in Appendix are given with independent double numeration.

2 The potential of a conducting axially symmetric body on the axis of an electric field

The potential of the electric field $\varphi(\mathbf{r})$, when it is disturbed by a conducting body, is equal to the sum $\varphi_0(\mathbf{r}) + u(\mathbf{r})$, where $\varphi_0(\mathbf{r})$ is the potential of the external initial field, and $u(\mathbf{r})$ is the potential generated by surface charges of the body. The potential $u(\mathbf{r})$ is determined by the formula

$$(1) \quad u(\mathbf{r}) = \frac{1}{4\pi\epsilon_0} \int_S \frac{\sigma(\mathbf{r}') dS}{|\mathbf{r} - \mathbf{r}'|},$$

where \mathbf{r}' is a coordinate on the surface S of the body, $\sigma(\mathbf{r}')$ is the surface charge density that provides the equipotentiality of the surface S . The

axially symmetric initial electric potential $\varphi_0(Z, X)$ induces the axially symmetric distribution of the surface charges $\sigma(z)$ on the surface of a conducting axially symmetric body. These charges generate the following potential on the axis of the electric field at a point s

$$(2) \quad u(s) = \frac{1}{2\varepsilon_0} \int_{z_1}^{z_2} \frac{h(z)}{\sqrt{(z-s)^2 + h^2(z)}} \sigma(z) dl(z),$$

where $dl(z)$ is an element of length of the generator line of the body of revolution, $h(z)$ is a radius of the cross-section of the body at a distance z from the origin, z_1 and z_2 are coordinates of the intersection of the axis of symmetry with the body surface. If the electric potential $u(s)$ compensates the initial potential $\varphi_0(s) = \varphi_0(s, 0)$ on the axis, then the total potential inside the body will be equal to zero not only on the axis, but also on the whole volume of the body, because every component of the potential satisfies the Laplace equation [2].

3 The charges on the surface of a conducting ball

For a conducting ball with a radius r , placed on the axis in the external electric field, equation (2) takes the form

$$(3) \quad \int_{-r}^r \frac{r \sigma(z)}{\sqrt{s^2 + r^2 - 2sz}} dz = -2\varepsilon_0 \varphi_0(s) \quad , \quad -r < s < r \quad .$$

Let us consider dimensionless variables $\xi = s/r$, $\eta = z/r$. Then equation (3) can be written down as

$$(4) \quad \int_{-1}^1 \frac{r \sigma(r\eta)}{\sqrt{\xi^2 + 1 - 2\xi\eta}} d\eta = -2\varepsilon_0 \varphi_0(r\xi) \quad , \quad -1 < \xi < 1.$$

If $\sigma(r\eta) = \eta^n$, where n is a natural number, then, integrating the left-hand side in (4), we obtain, using the mathematical induction, a polynomial of n degree also in the right-hand side depending on the variable ξ . Therefore when the right-hand side in equation (4) varies along the axis as a polynomial of n degree

$$-\varphi_0(r\xi) = \sum_{i=1}^{n+1} b'_i \xi^{i-1},$$

4 A CONDUCTING BALL IN AN AXIAL ELECTRIC FIELD

then $\sigma(r\eta)$ is also a polynomial of n degree

$$\sigma(r\eta) = \frac{2\varepsilon_0}{r} \sum_{j=1}^{n+1} c'_j \eta^{j-1},$$

with the coefficients c'_j that can be determined from the equation

$$\mathbf{F}\mathbf{c}' = \mathbf{b}',$$

where $\mathbf{c}' = (c'_1, \dots, c'_{n+1})^T$, $\mathbf{b}' = (b'_1, \dots, b'_{n+1})^T$, $\mathbf{F} = \{F_{ij}\}$, $i, j = 1, \dots, n+1$, and the matrix \mathbf{F} is an upper triangular one. The nonzero entries in this matrix are equal to

$$(5) \quad F_{ij} = \frac{1}{(i-1)!} \int_{-1}^1 \left(\frac{\partial^{i-1}}{\partial \xi^{i-1}} \frac{1}{\sqrt{\xi^2 + 1 - 2\xi\eta}} \right)_{\xi=0} \eta^{j-1} d\eta \quad .$$

Then the matrix $\mathbf{G} = \mathbf{F}^{-1} = \{G_{ij}\}$ will also be the upper triangular one.

Let the potential of an external electric field vary along the axis as a polynomial of n degree

$$(6) \quad -\varphi_0(s) = \sum_{i=1}^{n+1} b_i s^{i-1}.$$

Then the density of a surface charge is a polynomial of the same degree

$$(7) \quad \sigma(z) = \frac{2\varepsilon_0}{r} \sum_{j=1}^{n+1} c_j z^{j-1},$$

and the coefficients can be determined from the equation

$$(8) \quad \mathbf{F}\mathbf{R}\mathbf{c} = \mathbf{R}\mathbf{b} \quad .$$

Hence, the required coefficients can be found in the explicit form

$$(9) \quad \mathbf{c} = \mathbf{R}^{-1}\mathbf{G}\mathbf{R}\mathbf{b}$$

or

$$(10) \quad \mathbf{c} = \tilde{\mathbf{G}}\mathbf{b},$$

where $\mathbf{c} = (c_1, \dots, c_{n+1})^T$, $\mathbf{b} = (b_1, \dots, b_{n+1})^T$, $\mathbf{R} = \text{diag}\{1, r, \dots, r^n\}$, and entries of the matrix $\tilde{\mathbf{G}}$ are determined by a simple equality $\tilde{G}_{ij} = r^{j-i}G_{ij}$, $i, j = 1, \dots, n+1$.

The explicit form of entries of the matrix \mathbf{G} is found in Appendix (Property 8). Hence, the density of a surface charge in the explicit form can be found, when the potential of an external electric field on the axis is a polynomial of an arbitrary degree.

4 The full charge and dipole moment of a conducting ball

The full charge Q and dipole moment \mathcal{D} of a conducting ball with radius r on the axis of the axially symmetric electric field are determined by the formulas

$$(11) \quad Q = 2\pi \int_{-r}^r \sigma(z) h(z) dl(z) = 2\pi r \int_{-r}^r \sigma(z) dz,$$

$$(12) \quad \mathcal{D} = 2\pi r \int_{-r}^r z \sigma(z) dz.$$

If the potential of the external field is a linear function on the axis of symmetry $-\varphi_0(s) = b_1 + b_2 s$, then from (7) and (10) we obtain

$$(13) \quad \sigma(z) = \frac{2\varepsilon_0}{r} (c_1 + c_2 z) = \frac{2\varepsilon_0}{r} (G_{11}b_1 + G_{22}b_2 z).$$

From Property 8 (Appendix), which determines the entries of the inverse matrix \mathbf{G} , one can easily obtain $G_{11} = \frac{1}{2}$, $G_{22} = \frac{3}{2}$. Thus, from formulas (11)-(13) we obtain

$$(14) \quad Q = 4\pi\varepsilon_0 r b_1,$$

$$(15) \quad \mathcal{D} = 4\pi\varepsilon_0 r^3 b_2.$$

Theorem 1. *The full charge and dipole moment of a conducting ball are determined by formulas (14) and (15) in any external axially symmetric field that is harmonic inside the ball.*

Proof. Theorem is proved by induction. Assume that formulas (14) and (15) are valid provided that the potential $\varphi_0(s)$ is a polynomial of $n-1$ degree. This means that such equalities are valid for polynomials $\sigma(z)$ of $n-1$ degree, with coefficients that are determined from the solution of a system of linear equations of order n with an arbitrary right-hand side, where entries of the matrix \mathbf{F} are determined by equalities (5). We prove the validity of equalities (14) and (15), when the potential $\varphi_0(s)$, determined in (6), is a polynomial of n degree. In this case, the coefficients of polynomial (7) are determined from the system of linear equations (8) of order $n+1$, in which the matrix \mathbf{F} is the upper triangular one. Then the last component of the vector \mathbf{c} can be determined immediately from the equality

$$(16) \quad c_{n+1} = \frac{b_{n+1}}{F_{n+1,n+1}},$$

6 A CONDUCTING BALL IN AN AXIAL ELECTRIC FIELD

and the system of equations (8) can be written down as a system of linear equations of order n

$$(17) \quad \begin{aligned} \mathbf{F}^{(n)} \mathbf{R}^{(n-1)} \mathbf{c}^{(n-1)} &= \mathbf{R}^{(n-1)} \mathbf{b}^{(n-1)} - c_{n+1} r^n \mathbf{f}_{\bullet, n+1} = \\ \mathbf{R}^{(n-1)} [\mathbf{b}^{(n-1)} - c_{n+1} r^n (\mathbf{R}^{(n-1)})^{-1} \mathbf{f}_{\bullet, n+1}] & \end{aligned},$$

where

$\mathbf{F}^{(n)} = \{F_{ij}\}$, $i, j = 1, \dots, n$; $\mathbf{c}^{(n-1)} = (c_1, c_2, \dots, c_n)^T$; $\mathbf{b}^{(n-1)} = (b_1, b_2, \dots, b_n)^T$; $\mathbf{R}^{(n-1)} = \text{diag}\{1, r, \dots, r^{n-1}\}$; $\mathbf{f}_{\bullet, n+1} = (F_{1, n+1}, \dots, F_{n, n+1})^T$, and c_{n+1} is not just a required variable, but it is determined by entries of the matrix \mathbf{F} and the right-hand side \mathbf{b} .

Denote $\sigma^{(n-1)}(z) = \frac{2\varepsilon_0}{r} \sum_{j=1}^n c_j z^{j-1}$ and write equations (11) and (12) in the form

$$(18) \quad Q = Q^{(n-1)} + 4\pi\varepsilon_0 c_{n+1} r^{n+1} \int_{-1}^1 z^n dz,$$

$$(19) \quad \mathcal{D} = \mathcal{D}^{(n-1)} + 4\pi\varepsilon_0 c_{n+1} r^{n+2} \int_{-1}^1 z^{n+1} dz,$$

where

$$Q^{(n-1)} = 2\pi r \int_{-r}^r \sigma^{(n-1)}(z) dz, \quad$$

$$\mathcal{D}^{(n-1)} = 2\pi r \int_{-r}^r z \sigma^{(n-1)}(z) dz.$$

Since the induction assumption is valid for the system of equations (17) of order n , then

$$(20) \quad Q^{(n-1)} = 4\pi\varepsilon_0 r (b_1 - c_{n+1} r^n F_{1, n+1}),$$

$$(21) \quad \mathcal{D}^{(n-1)} = 4\pi\varepsilon_0 r^3 (b_2 - c_{n+1} r^{n-1} F_{2, n+1}).$$

From (5) we have

$$(22) \quad F_{1, n+1} = \int_{-1}^1 \eta^n d\eta,$$

$$(23) \quad F_{2,n+1} = \int_{-1}^1 \left(\frac{\partial}{\partial s} \frac{1}{\sqrt{1+s^2-2s\eta}} \right)_{s=0} \eta^n d\eta = \int_{-1}^1 \eta^{n+1} d\eta.$$

The proof of the theorem follows from formulas (18)-(23).

Corollary 1. *Formulas (14) and (15) are valid for any external harmonic fields that are not necessarily axially symmetric fields.*

Proof. The electric field, generated by a point charge is always axially symmetric for the ball, when the axis passes through its center and this charge. Then any distribution of point charges excites axial fields with axes passing through the center of the ball. Since Theorem 1 is also valid for the sum of electric fields with axes passing through the center of the ball and, since an electric field can always be presented as superposition of point charges, then Theorem 1 is valid for any electric field.

5 The dipole moments of higher orders of a conducting ball

Now let us discuss the deduction of the formula for the dipole moment of a conducting ball of m order, which is determined by the formula

$$\mathcal{D}_m = 2\pi r \int_{-r}^r z^m \sigma(z) dz.$$

Theorem 2. *The dipole moment of the ball of m order, when the potential of the external field $\varphi_0(s)$ on the axis of symmetry is a polynomial of n degree is equal to*

$$(24) \quad \mathcal{D}_m = 2\pi\epsilon_0 r^{m+1} \sum_{i=\delta, (2)}^{m+1} (2i-1) r^{i-1} F_{i,m+1} b_i, \quad ,$$

where $\delta = 1$ if m is even and $\delta = 2$ if m is odd.

In formula (24) and further the index “(2)” in the lower part of the sum symbol means that summation is carried out with a step equal to 2. Thus, in this case, the summation can be done only by odd or only by even values of the index i , depending on its first value.

Proof.

8 A CONDUCTING BALL IN AN AXIAL ELECTRIC FIELD

From (7) and Theorem 1 follows

$$\mathcal{D}_m = 2\pi r \int_{-r}^r z^m \sigma(z) dz = 4\pi\epsilon_0 \int_{-r}^r \sum_{j=1}^{n+1} c_j z^{m+j-1} dz = 4\pi\epsilon_0 r b'_1,$$

where b'_1 is the first component of the right-hand side in the system of equations

$$(25) \quad \begin{pmatrix} F_{11} & 0 & F_{13} & 0 & \dots & F_{1,M} \\ 0 & F_{22} & 0 & F_{24} & \dots & F_{2,M} \\ 0 & 0 & F_{33} & 0 & \dots & F_{3,M} \\ 0 & 0 & 0 & F_{44} & \dots & F_{4,M} \\ \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & \dots & F_{M,M} \end{pmatrix} \begin{pmatrix} 0 \\ \vdots \\ 0 \\ r^m c_1 \\ \vdots \\ r^{M-1} c_{n+1} \end{pmatrix} = \begin{pmatrix} b'_1 \\ r b'_2 \\ r^2 b'_3 \\ \vdots \\ r^{M-1} b'_M \end{pmatrix},$$

where $M = n + m + 1$. From (25) follows

$$(26) \quad \tilde{\mathbf{F}} \tilde{\mathbf{R}} \mathbf{c} = \mathbf{R} \mathbf{b}' \quad ,$$

where $\mathbf{b}' = (b'_1, \dots, b'_{n+1})^T$, $\tilde{\mathbf{R}} = \text{diag} \{r^m, r^{m+1}, \dots, r^{M-1}\}$,

$$\tilde{\mathbf{F}} = \begin{pmatrix} F_{1,m+1} & F_{1,m+2} & \dots & F_{1,M} \\ F_{2,m+1} & F_{2,m+2} & \dots & F_{2,M} \\ \dots & \dots & \dots & \dots \\ F_{n+1,m+1} & F_{n+1,m+2} & \dots & F_{n+1,M} \end{pmatrix}.$$

We express the vector \mathbf{b}' through the vector \mathbf{b} , using (9):

$$\tilde{\mathbf{F}} \tilde{\mathbf{R}} \mathbf{R}^{-1} \mathbf{G} \mathbf{R} \mathbf{b} = \mathbf{R} \mathbf{b}'.$$

Consequently,

$$(27) \quad r^m \tilde{\mathbf{F}} \mathbf{G} \mathbf{R} \mathbf{b} = \mathbf{R} \mathbf{b}' \quad .$$

Hence, the first component of the vector \mathbf{b}' is equal to the first line of the matrix $\mathbf{L} = \tilde{\mathbf{F}} \mathbf{G}$, multiplied by the vector $\mathbf{R} \mathbf{b}$ and the number r^m . We will prove that only $L_{1,\delta}, L_{1,\delta+2}, \dots, L_{1,m+1}$ are non-zero entries in this line.

Because the matrix \mathbf{G} is inverse to the matrix \mathbf{F} , the vectors $\mathbf{g}_{\bullet,j}$ if $j = m+2, \dots, n$, are orthogonal to the vectors $\mathbf{f}_{i,\bullet}$ when $i = 1, \dots, m+1$. From Lemma about a linear combination of vectors (see Appendix) it follows that the vector $\mathbf{f}' = (F_{1,m+1}, F_{1,m+2}, \dots, F_{1,M})^T$ is a linear combination of the vectors $\mathbf{f}_{\delta,\bullet}, \mathbf{f}_{\delta+2,\bullet}, \dots, \mathbf{f}_{m+1,\bullet}$ and, hence, it will be orthogonal to all the vectors $\mathbf{g}_{\bullet,j}$, $j = m+2, \dots, n$. Hence, $L_{1,l} = 0$ when $l > m+1$.

Let us find the first $m+1$ entries in the first line of the matrix \mathbf{L} . Because the vector \mathbf{f}' is a linear combination of the vectors $\mathbf{f}_{\delta,\bullet}, \mathbf{f}_{\delta+2,\bullet}, \dots, \mathbf{f}_{m+1,\bullet}$ with

the coefficients $\alpha_\delta, \alpha_{\delta+2}, \dots, \alpha_{m+1}$, then, by virtue of orthogonality of the vectors $\mathbf{f}_{i,\bullet}$ and $\mathbf{g}_{\bullet,l}$, when $i \neq l$ and $l \leq m+1$, we have

$$L_{1l} = (\mathbf{f}', \mathbf{g}_{\bullet,l}) = \sum_{i=\delta,(2)}^{m+1} \alpha_i (\mathbf{f}_{i,\bullet}, \mathbf{g}_{\bullet,l}) = \alpha_l,$$

where $\alpha_l = \frac{2l-1}{2} F_{l,m+1}$ (see formula (9.1) in Appendix). Then from (27) follows

$$b'_1 = \frac{r^m}{2} \sum_{i=\delta,(2)}^{m+1} (2i-1) r^{i-1} F_{i,m+1} b_i,$$

and

$$\mathcal{D}_m = 2\pi\epsilon_0 r^{m+1} \sum_{i=\delta,(2)}^{m+1} (2i-1) r^{i-1} F_{i,m+1} b_i.$$

6 The force acting on a conducting ball

The force \mathcal{F} , acting on the conductive ball with a radius r that is on the axis of the axially symmetric electric field, is equal to

$$\mathcal{F} = 2\pi \int_{-r}^r p(z) h(z) dh(z),$$

where $p(z)$ is electric pressure that is determined by the formula $p(z) = -\frac{\sigma^2(z)}{2\epsilon_0}$ [1]. From here it follows that

$$(28) \quad \mathcal{F} = \frac{\pi}{\epsilon_0} \int_{-r}^r z \sigma^2(z) dz.$$

Theorem 3. *The force \mathcal{F} , acting on the conducting ball with a radius r , located on the axis of the axially symmetric electric field with a potential varying along the axis as polynomial of n degree (see formula (6)), is uniquely defined by coefficients of this polynomial b_1, \dots, b_{n+1} and is equal to*

$$\mathcal{F} = 4\pi\epsilon_0 \sum_{i=1}^n i r^{2i-1} b_i b_{i+1}.$$

Proof. The theorem is proved by induction.

10 A CONDUCTING BALL IN AN AXIAL ELECTRIC FIELD

Let $n = 1$. Then from (7) and (10) follows

$$\sigma(z) = \frac{2\varepsilon_0}{r} (c_1 + c_2 z) = \frac{2\varepsilon_0}{r} (G_{11}b_1 + G_{22}b_2 z).$$

From Property 8 (Appendix), which determines the entries of the inverse matrix \mathbf{G} , it follows that $G_{11} = \frac{1}{2}$, $G_{22} = \frac{3}{2}$. Then

$$\mathcal{F} = \frac{4\pi\varepsilon_0}{r^2} \int_{-r}^r z \left(\frac{1}{2}b_1 + \frac{3}{2}b_2 z \right)^2 dz = 4\pi\varepsilon_0 r b_1 b_2.$$

Assume that the statement of the theorem is valid for $k = n - 1$ and we will prove its validity for $k = n$.

Since the integral of an odd function from $-r$ to r equals zero, then

$$(29) \quad \begin{aligned} & \int_{-r}^r z \left(\sum_{j=1}^{n+1} c_j z^{j-1} \right)^2 dz = \\ & \int_{-r}^r z \left(\sum_{j=1}^n c_j z^{j-1} \right)^2 dz + 2c_{n+1} \int_{-r}^r z^{n+1} \left(\sum_{j=1}^n c_j z^{j-1} \right) dz. \end{aligned}$$

From Theorem 2 it follows that

$$(30) \quad \begin{aligned} & \int_{-r}^r z^{n+1} \left(\sum_{j=1}^n c_j z^{j-1} \right) dz = \int_{-r}^r z^{n+1} \left(\sum_{j=1}^{n+1} c_j z^{j-1} \right) dz = \\ & \frac{r^{n+2}}{2} \sum_{i=\delta, (2)}^n (2i-1) r^{i-1} F_{i, n+2} b_i \end{aligned}$$

Let us consider the first summand in the right-hand side of (29). As in the proof of Theorem 1, we note that the system of equations (8) of order $n + 1$ is equivalent to the system of equations (17) of order n , where the last component of the vector \mathbf{c} is determined in (16). Because the i -th component of the vector in the square brackets in (17) equals $b'_i = b_i - c_{n+1} r^{n-i+1} F_{i, n+1}$, then, by the induction hypothesis,

$$\frac{\pi}{\varepsilon_0} \int_{-r}^r z \sigma_{n-1}^2(z) dz = 4\pi\varepsilon_0 \sum_{i=1}^{n-1} i r^{2i-1} b'_i b'_{i+1},$$

where $\sigma_{n-1}(z) = \frac{2\varepsilon_0}{r} \sum_{j=1}^n c_j z^{j-1}$. In view of the fact that $F_{i,n+1}F_{i+1,n+1} = 0$ for any natural numbers i and n (Property 2, Appendix)

(31)

$$\begin{aligned} \int_{-r}^r z \left(\sum_{j=1}^n c_j z^{j-1} \right)^2 dz = \\ \sum_{i=1}^{n-1} i r^{2i+1} (b_i - c_{n+1} r^{n-i+1} F_{i,n+1}) (b_{i+1} - c_{n+1} r^{n-i} F_{i+1,n+1}) = \\ \sum_{i=1}^{n-1} i r^{2i+1} b_i b_{i+1} - \sum_{i=2}^n (i-1) c_{n+1} r^{n+i+1} F_{i-1,n+1} b_i - \sum_{i=1}^{n-1} i c_{n+1} r^{n+i+1} F_{i+1,n+1} b_i. \end{aligned}$$

We continue the proof of the induction hypothesis for $k = n$, considering the cases of odd and even values of n , separately.

1. Let n be an odd number. Then by virtue of (29) - (31), we have

$$(32) \quad \int_{-r}^r z \left(\sum_{j=1}^{n+1} c_j z^{j-1} \right)^2 dz = \sum_{i=1}^{n-1} i r^{2i+1} b_i b_{i+1} + S_n^{(1)},$$

where

$$\begin{aligned} S_n^{(1)} / c_{n+1} = r^{n+2} \sum_{i=1,(2)}^n (2i-1) r^{i-1} F_{i,n+2} b_i - \\ - \sum_{i=3,(2)}^n (i-1) r^{n+i+1} F_{i-1,n+1} b_i - \sum_{i=1,(2)}^{n-2} i r^{n+i+1} F_{i+1,n+1} b_i. \end{aligned}$$

From here it follows that

(33)

$$\begin{aligned} S_n^{(1)} / c_{n+1} = \\ r^{n+2} \{ (F_{1,n+2} - F_{2,n+1}) b_1 + [(2n-1) F_{n,n+2} - (n-1) F_{n-1,n+1}] r^{n-1} b_n \} + \\ \sum_{i=3,(2)}^{n-2} r^{n+i+1} b_i [(2i-1) F_{i,n+2} - (i-1) F_{i-1,n+1} - i F_{i+1,n+1}]. \end{aligned}$$

Since $F_{2,n+1} = F_{1,n+2}$ for any values of n (Property 5, Appendix), then the first summand in the right-hand side of (33) is equal to zero. From the recurrence relation for entries of the matrix \mathbf{F} (Property 3, Appendix), we have

$$(34) \quad F_{i+1,n+1} = \frac{2i-1}{i} F_{i,n+2} - \frac{i-1}{i} F_{i-1,n+1}.$$

12 A CONDUCTING BALL IN AN AXIAL ELECTRIC FIELD

Hence, the third summand in the right-hand side of (33) is also equal to zero. Since the last component of the vector \mathbf{c} can be found from (16), then

$$S_n^{(1)} = \frac{b_n b_{n+1}}{F_{n+1, n+1}} r^{2n+1} [(2n-1) F_{n, n+2} - (n-1) F_{n-1, n+1}].$$

Using Lemma 1 and Lemma 2 from Appendix, we obtain

$$S_n^{(1)} = r^{2n+1} b_n b_{n+1} \left[\frac{2^{n+1} (2n-1) [(n+1)!]^2}{(2n+2)!} - \frac{2^n (n-1) [(n!)^2]}{(2n)!} \right] \frac{(2n+2)!}{2^{n+2} n! (n+1)!} = \frac{r^{2n+1} b_n b_{n+1}}{2} [(2n-1)(n+1) - (n-1)(2n+1)] = n r^{2n+1} b_n b_{n+1}.$$

Thus, from (32) we have

$$(35) \quad \int_{-r}^r z \left(\sum_{j=1}^{n+1} c_j z^{j-1} \right)^2 dz = \sum_{i=1}^n i r^{2i+1} b_i b_{i+1}.$$

2. Let the number n be even. Then from (29) - (31) we obtain

$$\int_{-r}^r z \left(\sum_{j=1}^{n+1} c_j z^{j-1} \right)^2 dz = \sum_{i=1}^{n-1} i r^{2i+1} b_i b_{i+1} + S_n^{(2)},$$

where

$$S_n^{(2)} / c_{n+1} = r^{n+2} \sum_{i=2, (2)}^n (2i-1) r^{i-1} F_{i, n+2} b_i - \sum_{i=2, (2)}^n (i-1) r^{n+i+1} F_{i-1, n+1} b_i - \sum_{i=2, (2)}^{n-2} i r^{n+i+1} F_{i+1, n+1} b_i.$$

From here follows

$$S_n^{(2)} / c_{n+1} = r^{2n+1} \{ [(2n-1) F_{n, n+2} - (n-1) F_{n-1, n+1}] b_n \} + \sum_{i=2, (2)}^{n-2} r^{n+i+1} b_i [(2i-1) F_{i, n+2} - (i-1) F_{i-1, n+1} - i F_{i+1, n+1}].$$

In view of (34), the second summand in the right-hand side of the equality is equal to zero and is easy to see that in this case $S_n^{(2)} = S_n^{(1)}$. Thus, equation (35) is valid for any values of the index n . Then from (7), (28) and (35) we obtain

$$\mathcal{F} = \frac{\pi}{\varepsilon_0} \frac{4\varepsilon_0^2}{r^2} \int_{-r}^r z \left(\sum_{j=1}^{n+1} c_j z^{j-1} \right)^2 dz = 4\pi\varepsilon_0 \sum_{i=1}^n i r^{2i-1} b_i b_{i+1},$$

and the theorem is proved.

APPENDIX

Properties of the matrix of moments of the Legendre polynomials

.1 The matrix \mathbf{F} is the matrix of moments of the Legendre polynomials

Indeed, as the function $\frac{1}{\sqrt{1-2\xi\eta+\xi^2}}$ is the generating function for the Legendre polynomials $P_n(\eta)$, i.e. $\sum_{n=0}^{\infty} P_n(\eta) \xi^n = \frac{1}{\sqrt{1-2\xi\eta+\xi^2}}$, then

$$\left(\frac{\partial}{\partial \xi^{i-1}} \frac{1}{\sqrt{1-2\xi\eta+\xi^2}} \right)_{\xi=0} = (i-1)! P_{i-1}(\eta)$$

for any natural values i . Hence,

$$F_{ij} = \int_{-1}^1 P_{i-1}(\eta) \eta^{j-1} d\eta \quad . \quad (1.1)$$

.2 The odd upper diagonals of the matrix \mathbf{F} consist of zero entries, i.e. $F_{ij} = 0$ if $i + j$ is an odd number

Obviously, the polynomial $P_n(\eta)$ is even if the number n is even, and is odd otherwise. This statement is easily proved by induction with allowance for the recurrence formula for the Legendre polynomials

$$P_{n+1}(\eta) = \frac{2n+1}{n+1} \eta P_n(\eta) - \frac{n}{n+1} P_{n-1}(\eta) . \quad (2.1)$$

Since the integral of an odd function is equal to zero, then Property 2 is proved.

.3 The entries of the matrix \mathbf{F} satisfy the recurrence relation

$$F_{ij} = \frac{2i-3}{i-1} F_{i-1,j+1} - \frac{i-2}{i-1} F_{i-2,j} \quad (3.1)$$

when $i \geq 3$.

This statement follows from formulas (1.1) and (2.1).

.4 The matrix \mathbf{F} is an upper triangular one, i.e. $F_{ij} = 0$, when $i > j$.

The polynomial $P_i(\eta)$ is orthogonal to 1, η , $P_2(\eta)$ when $i > 2$. Hence $P_i(\eta) \perp \eta^2$. Continuing this argument we can conclude that if $P_i(\eta)$ is orthogonal to all η^k when $k < j$, then, because $P_i(\eta) \perp P_j(\eta)$, we have $P_i(\eta) \perp \eta^j$. From here the proof of Property 4 follows.

.5 The entries of the matrix \mathbf{F}

The entries of the first and second lines of the matrix \mathbf{F} can be found from formula (5):

$$F_{1,j} = \frac{2}{j} \quad \text{if } j \text{ is odd}, \quad F_{2,j} = \frac{2}{j+1} \quad \text{if } j \text{ is even.} \quad (5.1)$$

An arbitrary entry of the matrix can be found using the Rodrigues formula for the Legendre polynomials

$$P_n(\eta) = \frac{1}{2^n n!} \frac{d^n}{d\eta^n} (\eta^2 - 1)^n.$$

Then, in view of (1.1),

$$F_{ij} = \frac{1}{2^{i-1} (i-1)!} \sum_{k=0}^{i-1} (-1)^k C_{i-1}^k \int_{-1}^1 \eta^{j-1} \frac{d^{i-1}}{d\eta^{i-1}} \eta^{2(i-k-1)} d\eta.$$

From here it follows that

$$F_{ij} = \frac{1}{2^{i-2}} \sum_{k=0}^{[(i-1)/2]} (-1)^k \frac{(2i-2k-2)!}{k! (i-k-1)! (i-2k-1)! (i-2k-1+j)!}. \quad (5.2)$$

Lemma 1. The diagonal entries of the matrix \mathbf{F} are the following:

$$F_{ii} = \frac{2^{i+1} i! (i-1)!}{(2i)!}.$$

Proof. Lemma is proved by induction. For $i = 1$, this statement is obviously valid. We assume that it is true for $i = k$. From recurrence formula (3.1) and Property 4 it follows that

$$F_{k+2,k} = \frac{2k+1}{k+1} F_{k+1,k+1} - \frac{k}{k+1} F_{kk} = 0.$$

Hence,

$$F_{k+1,k+1} = \frac{k}{(2k+1)} \frac{2^{k+1}k!(k-1)!}{(2k)!} = \frac{2^{k+1}(k!)^2}{(2k+1)!} = \frac{2^{k+2}k!(k+1)!}{(2k+2)!}.$$

Lemma 2. *The entries of the second upper diagonal of the matrix \mathbf{F} are equal to*

$$F_{i-2,i} = \frac{2^{i-1}[(i-1)!]^2}{(2i-2)!}, \quad i \geq 3.$$

Proof. Lemma is proved by induction. For $i = 3$, this statement is valid (see (5.1)). Assume that it is true for $i = k$. Then from (3.1) and Lemma 1 it follows that

$$F_{kk} = \frac{2^{k+1}k!(k-1)!}{(2k)!} = \frac{2k-3}{k-1}F_{k-1,k+1} - \frac{k-2}{k-1}F_{k-2,k}.$$

From here

$$F_{k-1,k+1} = \frac{2^{k+1}(k-1)k!(k-1)!}{(2k-3)(2k)!} + \frac{2^{k-1}(k-2)[(k-1)!]^2}{(2k-3)(2k-2)!} = \frac{2^k(k!)^2}{(2k)!}.$$

Hence, the induction hypothesis also holds for $i = k + 1$.

.6 Lemma about a linear combination of lines

Lemma 3. *The vector $\mathbf{f}' = (F_{1,m+1}, F_{1,m+2}, \dots, F_{1,m+n+1})^T$ is a linear combination of the vectors $\mathbf{f}_{\delta,\bullet}, \mathbf{f}_{\delta+2,\bullet}, \dots, \mathbf{f}_{m+1,\bullet}$, where $\delta = 1$ if m is even, and $\delta = 2$ if m is odd.*

Here we denoted $\mathbf{f}_{k,\bullet} = (F_{k1}, F_{k2}, \dots, F_{k,n+1})^T$ for any natural k .

Proof.

We write down formula (5.2) in the form

$$F_{ij} = \frac{1}{2^{i-1}} \sum_{k=\delta,(2)}^i (-1)^{(i-k)/2} \frac{(i-k-2)!}{(k-1)!((i-k)/2)!((i+k)/2-1)!} \frac{2}{(j+k-1)} \quad (6.1)$$

where $\delta = 1$ if i is odd, and $\delta = 2$ if i is even.

In formula (6.1) and further on, the index “(2)” in the lower part of the sum symbol means that summation is carried out with a step equal to 2.

16 A CONDUCTING BALL IN AN AXIAL ELECTRIC FIELD

Then, taking into account (5.1), formula (6.1) can be written as

$$F_{ij} = \sum_{k=\delta, (2)}^i \beta_{ki} F_{1,j+k-1}, \quad (6.2)$$

where

$$\beta_{ki} = \frac{1}{2^{i-1}} (-1)^{(i-k)/2} \frac{(i+k-2)!}{(k-1)! ((i-k)/2)! ((i+k)/2-1)!} \quad (6.3)$$

If the vector \mathbf{f}' is a linear combination of the vectors $\mathbf{f}_{\delta, \bullet}, \mathbf{f}_{\delta+2, \bullet}, \dots, \mathbf{f}_{m+1, \bullet}$, then its j -th component must be equal to the j -th element of a linear combination of the vectors $\sum_{k=\delta, (2)}^{m+1} \alpha_k \mathbf{f}_{k, \bullet}$:

$$F_{1,j+m} = \sum_{k=\delta, (2)}^{m+1} \alpha_k \sum_{l=\delta, (2)}^k \beta_{lk} F_{1,j+l-1} = \sum_{l=\delta, (2)}^{m+1} F_{1,j+l-1} \sum_{k=l, (2)}^{m+1} \alpha_k \beta_{lk}. \quad (6.4)$$

We choose the coefficients of a linear combination $\alpha_\delta, \alpha_{\delta+2}, \dots, \alpha_{m+1}$ in such a way that equality (6.4) be valid for any values j , where $j = \delta, \delta+2, \dots, n+1$. For the next values of j in the interval from 1 up to $n+1$, equality (6.4) is automatically holds, because of the zero entries of the matrix (Property 2). Then the factors before the values $F_{1,j+l-1}$ in (6.4) must be equal to zero when $l = \delta, \delta+2, \dots, m-1$ and $\alpha_{m+1} \beta_{m+1} = 1$. Therefore, we obtain the following system of linear equations with a square matrix having the dimension $\left(\frac{m+3-\delta}{2}\right) \times \left(\frac{m+3-\delta}{2}\right)$:

$$\begin{pmatrix} 1 & \beta_{\delta, \delta+2} & \beta_{\delta, \delta+4} & \dots & \beta_{\delta, m+1} \\ 0 & \beta_{\delta+2, \delta+2} & \beta_{\delta+2, \delta+4} & \dots & \beta_{\delta+2, m+1} \\ 0 & 0 & \beta_{\delta+4, \delta+4} & \dots & \beta_{\delta+4, m+1} \\ \dots & \dots & \dots & \dots & \dots \\ 0 & \dots & \dots & \dots & \beta_{m+1, m+1} \end{pmatrix} \begin{pmatrix} \alpha_\delta \\ \alpha_{\delta+2} \\ \alpha_{\delta+4} \\ \vdots \\ \alpha_{m+1} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ \vdots \\ 0 \\ 1 \end{pmatrix} \quad (6.5)$$

This is a system of equations with a triangular matrix which has nonzero entries on the diagonal; hence it has the unique solution. Consequently, $\alpha_\delta, \alpha_{\delta+2}, \dots, \alpha_{m+1}$ are the required coefficients of expansion of the vector \mathbf{f}' to the vectors $\mathbf{f}_{\delta, \bullet}, \mathbf{f}_{\delta+2, \bullet}, \dots, \mathbf{f}_{m+1, \bullet}$.

Corollary: The system of equations (6.5) can be written down in a general form:

$$\mathbf{B}\mathbf{a} = \mathbf{e} \quad , \quad (6.6)$$

where $\mathbf{a} = (\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_{m+1})^T$, $\mathbf{e} = (0, 0, \dots, 0, 1)^T$, and the matrix \mathbf{B}

$$\mathbf{B} = \begin{pmatrix} 1 & 0 & \beta_{13} & 0 & \beta_{15} & 0 & \dots & \beta_{1,m+1} \\ 0 & 1 & 0 & \beta_{24} & 0 & \beta_{26} & \dots & \beta_{2,m+1} \\ 0 & 0 & \beta_{33} & 0 & \beta_{35} & 0 & \dots & \beta_{3,m+1} \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ 0 & 0 & 0 & 0 & 0 & 0 & \dots & \beta_{m+1,m+1} \end{pmatrix}, \quad (6.7)$$

where $\beta_{ij} = 0$ if $i + j$ is an odd number, and m is an arbitrary natural number.

.7 Lemma about orthogonality

Lemma 4. *For the matrix \mathbf{F} from (5) and \mathbf{B} from (6.7) when $m = n$, the following equality holds:*

$$\mathbf{FB} = \mathbf{D}, \quad (7.1)$$

where \mathbf{D} is a diagonal matrix with diagonal entries equal to

$$D_{ii} = \frac{2}{2i-1}. \quad (7.2)$$

Proof.

As the matrices \mathbf{F} and \mathbf{B} are the upper triangular ones, then their product will be the same matrix. We will prove that all the entries of the matrix \mathbf{D} that are above the diagonal will be equal to zero. We multiply the i -th line of the matrix \mathbf{F} by the j -th row of the matrix \mathbf{B} , $i < j$. If $i + j$ is odd, then the product is equal to zero because of Property 2 and the same form of the matrix \mathbf{B} . Let $i + j$ be even. Then

$$D_{ij} = (\mathbf{f}_{i,\bullet}, \mathbf{\beta}_{\bullet,j}) = \sum_{l=0,(2)}^{j-i} \beta_{i+l,j} F_{i,i+l}.$$

By virtue of (6.2)

$$D_{ij} = \sum_{l=0,(2)}^{j-i} \beta_{i+l,j} \sum_{k=\delta,(2)}^i \beta_{ki} F_{1,i+l+k-1} = \sum_{k=\delta,(2)}^i \beta_{ki} \sum_{l=0,(2)}^{j-i} \beta_{i+l,j} F_{1,i+l+k-1},$$

where $\delta = 1$ if i, j are odd, and $\delta = 2$ if i, j are even.

On the other hand, by virtue of Property 4,

$$F_{jk} = \sum_{m=\delta,(2)}^j \beta_{mj} F_{1,k+m-1} = \sum_{m=\delta,(2)}^{i-2} \beta_{mj} F_{1,k+m-1} + \sum_{l=0,(2)}^{j-i} \beta_{i+l,j} F_{1,i+l+k-1} = 0.$$

Then

$$\begin{aligned}
 D_{ij} = & - \sum_{k=\delta,(2)}^i \beta_{ki} \sum_{m=\delta,(2)}^{i-2} \beta_{mj} F_{1,k+m-1} = \\
 & - \sum_{m=\delta,(2)}^{i-2} \beta_{mj} \sum_{k=\delta,(2)}^i \beta_{ki} F_{1,k+m-1} = - \sum_{m=\delta,(2)}^{i-2} \beta_{mj} F_{im} = 0
 \end{aligned}$$

The latter equality is carried out once again because of Property 4. Hence, the matrix \mathbf{D} is a diagonal one. To find the values of diagonal entries of this matrix, we use Lemma 1 and formula (6.3). In this case

$$D_{ii} = F_{ii}\beta_{ii} = \frac{2^{i+1}i!(i-1)!}{(2i)!} \frac{1}{2^{i-1}} \frac{(2i-2)!}{[(i-1)!]^2} = \frac{2}{2i-1} ,$$

and Lemma is proved.

.8 The inverse matrix entries

The entries of the inverse matrix $\mathbf{G} = \mathbf{F}^{-1}$ can be determined from equation (7.1) with allowance for (6.3) and (7.2). Really, $\mathbf{G} = \mathbf{B}\mathbf{D}^{-1}$, consequently,

$$G_{ij} = (-1)^{(j-i)/2} \frac{1}{2^j} \frac{(2j-1)(j+i-2)!}{(i-1)!((j-i)/2)!((j+i)/2-1)!} , \quad (8.1)$$

if $i+j$ is even, and $i \leq j$. Otherwise $G_{ij} = 0$.

.9 The coefficients of expansion of the vector \mathbf{f}' by lines of the matrix \mathbf{F}

The values of required coefficients $\alpha_1, \alpha_2, \alpha_3, \dots, \alpha_{m+1}$ are determined from the solution to the system of linear equations (6.6). From (7.1) it follows that $\mathbf{B} = \mathbf{G}\mathbf{D}$. Then from $\mathbf{G}\mathbf{D}\mathbf{a} = \mathbf{e}$ follows $\mathbf{D}\mathbf{a} = \mathbf{F}\mathbf{e}$ and, as $\mathbf{F}\mathbf{e}$ is the last row of the matrix \mathbf{F} and the diagonal entries of the matrix \mathbf{D} are determined in (7.2), then

$$\alpha_i = \frac{2i-1}{2} F_{i,m+1} . \quad (9.1)$$

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